Reasoning about Cardinal Directions Using Grids as Qualitative Geographic Coordinates*

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Abstract. In this article we propose a calculus of qualitative geographic coordinates which allows reasoning about cardinal directions on grid-based reference systems in maps. Grids in maps can be considered as absolute reference systems. The analysis reveals that the basic information coded in these reference systems is ordering information. Therefore, no metric information is required. We show that it is unnecessary to assume a coordinate system based on numbers in order to extract information like a point *P* is further north than a point *Q*. We investigate several grids in maps resulting from different types of projections. In addition, a detailed examination of the north arrow is given since it supplies a grid with ordering information. On this basis, we provide a general account on grids, their formalization and the inferences about cardinal directions drawn using qualitative geographic coordinates.

Keywords: Qualitative Spatial Reasoning, Map Projections, Geometry, Geography, Axiomatics, Inferences, Spatial Structure

1 Introduction

Geographic maps can be utilized to reason about cardinal directions between objects in the world. Cardinal directions in maps can be treated in a quantitative and a qualitative manner. The quantitative use of cardinal directions is employed in technical map use, for example in nautical navigation. As we are concerned with the use of maps by humans we focus on a qualitative approach. An overview and a classification of qualitative approaches applied to cardinal directions in geographic space can be found in Frank (1992). Our main objective is to specify conditions when a point *P* is further north (south/west/east) than a point *Q* in a simple geometric way without using or referring to analytical geometry.

General approaches describing spatial configurations in a qualitative way (cf. the 'double cross' calculus, Freksa 1992) can be used to relate cartographic entities in maps. But this kind of approaches neglects some specific properties of geographic maps. In particular, neither the different kinds of geometric information due to

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cartographic projection characteristics are explicitly modeled, nor the differences between cardinal directions on the map and those on the earth are considered. The relations 'further north/south' and 'further west/east' on the earth are asymmetric. 'Further north/south' constitute total linear bounded orders whereas 'further west/east' form cyclic orders. This results from the fact that the earth has a North Pole and a South Pole but does not have a West Pole or an East Pole. Since a map is bounded this asymmetry is abolished as long as the map does not contain any of the two poles. Hence, on maps which do not contain poles the relations 'further west/east' are also total linear bounded orders and all relations concerning cardinal directions are consequently structurally equal. This leads to symmetric descriptions of both pairs in the formal characterization.

The geographic maps considered here are supplied with a graticule which is a grid representing the meridians and the parallels of latitude. The grid lines are ordered either by the coordinates on the map boundary or by a north arrow. If the map boundary is supplied with coordinates only the precedence information of the coordinates but not their numerical value is required to order grid lines. However, the north arrow can be interpreted in two ways: As a map feature deciding for any two positions on a meridian which one is further north or as a geometric entity specifying for any pair of parallels of latitude which one is further north.

Grids are absolute reference systems because the spatial relation of two object positions is independent of an observer's perspective and of intrinsic properties of the related objects. Since the grid lines are ordered, the main objective is to show how to use grid lines to reason about cardinal relations. Therefore, we propose a calculus of qualitative geographic coordinates that can be embedded in the framework of ordering geometry (cf. Schlieder 1995, Eschenbach & Kulik 1997). The qualitative geographic coordinates are given by the grid lines and are supplied with an order by the north arrow or the ordering information of coordinates on a map boundary.

To develop the geometric framework which characterizes the spatial information given by cardinal directions we employ the axiomatic method. An axiomatic system does not define basic terms like 'point', 'grid line' or 'between'. Instead, it constitutes a system of constraints which determines the properties of these basic terms and specifies their relations. Therefore, an axiomatic specification of the graticule characterizes the underlying spatial structure of the map.

1.1 Reference Systems in Maps

Since we consider grids as absolute reference systems we give a short overview about reference systems. Reference systems are essential in contexts in which the same spatial constellation can be described from different 'perspectives'. Therefore, different descriptions are used for the same spatial constellation of objects in maps. Which reference systems are used in maps or a map interpreting context and which features constitute these reference systems? In Fig. 1 some geographic entities are depicted. The spatial relations of these entities depend on the applied reference system. According to Levinson (1996) who distinguishes three different types of reference systems—relative, intrinsic, and absolute ones—there are the following descriptions for two cities *P* and *Q*. (1) 'City *P* is to the left of city *Q*'. This statement is valid with respect to a relative reference system used in map-user interaction where a viewer is needed to supply an origin. (2) 'City *P* is on the right-side of the river'. This statement is true even if the map is rotated because of intrinsic features of the river. (3) 'City *P* is further north than city *Q*'. This description is valid using the absolute reference system of the map.

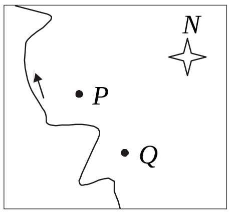


Fig. 1. A schematic map depicting two cities (*P* and *Q*) and a river

Absolute reference systems have a fixed direction which is—in the case of maps—provided by the North Pole. This direction is specified for every point on a map. The orientation of a map in the world is irrelevant. The description of spatial constellations is independent of a point of view and the relation between the reference object and the localized object is binary because only the position of the two objects are required (cf. Levinson 1996). The constituting elements of an absolute reference system in maps are the graticule and the north arrow.

2 Features of the Graticule and the North Arrow

2.1 Characteristics of the Graticule in Maps

Grids on the basis of geographic coordinates show a great variety in their form. Some maps have rectangular boundaries others have more elliptical outlines. The visualized grid structure consists of rectilinear parallels of latitude or meridians or have various combinations of curved graticule lines. We will give a short overview of the main differentiating characteristics of map projections and describe their specific mapping features.

A map projection is a mapping from a spherical surface—in this case the surface of the earth—to a planar medium. This comprises the reduction of one dimension from a three-dimensional sphere to a two-dimensional plane. Therefore, all projections must have distortions of at least one of the following qualities: angles, distances, or areas. On the map the principal scale—denoted as a fraction like 1/10.000 in the margin of the map—can only be preserved at certain points or along certain lines. They are known as points or lines of zero distortion. Particular scales which describe the scale at a certain point on the map, vary throughout the map according to position and direction. The following features characterize map projections sufficiently: the property, the class and the aspect (cf. Maling 1992; Robinson et al. 1995).

The Property of Map Projections. The following qualities determine the property of a projection: conformality (angle-preserving), equivalence (area-preserving), and equidistance (distance-preserving). These properties are closely related with the grid structure and they determine which kind of information can be inferred from a map. No map projection can preserve all of the properties and therefore has its characteristic pattern of distortion.

A *conformal projection* preserves angles. The intersections of the graticule between meridians and parallels of latitude are right-angled in the world and hence they must be right-angled in the map, for example the Mercator Projection. Due to this property this kind of projections is used in navigational, topographical or military maps. Conformality can only be obtained at the expanse of increasing particular scale. Areas at the margins of a map are therefore depicted much larger than they are, compared with central parts of the map. Because of these increasing distortions towards the margins these projections are not useful for the mapping of large areas like continents.

An *equivalent projection* maintains the content of areas. This means that the correspondence relation between the area in the world and the area in a map remains the same throughout the map. This can only be achieved with considerable distortions of form and distance. This kind of projections is used for depictions of statistical data. Maintaining the area prevents the interpreter of the map from e.g. drawing a false conclusion about the density of a distribution of goods, resources, people etc. visualized as symbols.

Equidistant projections preserve the principal scale throughout the map in a certain direction. This means that a given scale is true along certain lines. Usually this is the scale along the meridians. The result of this mapping is then that all parallels of latitude intersect all the meridians at a separation corresponding to the arc distance between the parallels on the globe. An equidistant projection is incompatible with conformality and equivalence. The usefulness is most important when a comprising projection between a conformal and an equal-area one is needed.

The Classes of Map Projections. The classes of map projections are specified due to the shape of the projection plane. They can be divided in azimuthal, cylindrical, and conical ones. Each class results in a characteristic pattern of the graticule.

Cylindrical projections result from projecting a spherical surface onto a tangent cylinder. There is one single line of zero distortion, the standard line. Scale is true only along this central line (the equator for normal, the central meridian for transverse, and a selected line for oblique) or along two lines equidistant from the central line in the case of a secant cylinder. The resulting graticule of this kind of projection is rectangular for the normal aspect. Scale distortions increase near points 90 degrees from the central line.

Conic projections result from the transformation from the sphere to the plane through the medium of a cone wrapped around the globe. The line or zero distortion corresponds to the circle of contact which is represented on the map by a circular arc in the normal aspect. The outline of the hemispherical map is fan-shaped.

Azimuthal projections result from a transformation to a projection plane that is tangential to the generating globe. The point of zero distortion corresponds to the point where the two surfaces meet. Such projections have the common property that all angles, azimuths in the general case, are correctly represented at the common point.

From the point of zero distortion at the center of the circle, the particular scales increase outwards in all directions.

The Aspect of a Map Projection. The aspect of a map projection denotes the position of the projection plane for the classes of map projections. Three main types—normal, transverse, and oblique—can be differentiated.

The *normal aspect* always provides the simplest graticule and calculations. In case of the azimuthal projection it touches one of the poles, for the cylindrical or conical projection the axis of the projection plane coincides with the axis of the ellipsoid.

We speak of a *transverse aspect* of a projection when the axis of the ellipsoid and the axis of the projections are perpendicular to each other in case of cylindrical or conical projections. The results of these projections are patterns of distortions rotated by 90° with respect to the normal aspect. However, a rectilinear structure of the graticule is not found anymore.

There are countless versions of map projection in the *oblique aspect*. The angle between the axis of the earth and the axis of the projection is arbitrary. The patterns of the graticule resulting from this kind of projections are the most complicated ones in the sense that the resulting form of the graticule is not rectilinear or even asymmetric.

The discussion of the main characteristics of map projections has shown that there are many factors that determine the form of the graticule. It is obvious now that no projection from a sphere to a plane can be at the same time conformal (angle-preserving), equivalent (area-preserving), or equidistant. On the other hand, the graticule can be used to reason about cardinal directions in each of these projections. Under a representation theoretic point of view (Palmer 1978) we can say that at least this quality is maintained in the mapping from the world to the map. This leads to the conclusion that information about cardinal directions must be coded in a sparser geometry than the ones which require information about distances or angles. Our analysis reveals that at least ordering information is needed to provide a suitable basis for inferences about cardinal directions in maps. The formalism is developed in sections 3 and 4.

2.2 The Graticule and the North Arrow

The north arrow is a map feature with two purposes: Supplying maps with a north direction and distinguishing meridians from parallels of latitude. Because of the discrepancy between a straight north arrow and curved meridians, for example in small-scale maps, the information that the north arrow implies becomes underspecific. Distinguishing two possibilities we obtain the following cases for an initial approach to fix the information provided by the north arrow: First, the north arrow determines for any two positions on a meridian which one is further north, i.e. it defines a precedence relation on each meridian. Second, the north arrow orders at least two parallels of latitude, i.e. specifies for any pair of parallels of latitude the one further north. The simplest condition for the first case is a map with a rectangular grid of parallels of latitude and meridians parallel to the map boundary. In this case the north arrow is parallel to the meridians. We can establish a simple correspondence between the north direction, the north arrow, the meridians, and the boundaries of the map: The north arrow provides ordering information of the kind ‘meridian g connects two map boundaries and leads from south boundary to north boundary when following the direction given by the north arrow'. Even if the map is not oriented to the north or to one of the other cardinal directions we can transfer the ordering information of the north arrow to the meridians using the parallelity between them and the north arrow. For curved meridians we have to make the assumption that in general the north arrow does not intersect the meridians. Hence, we can distinguish on class of grid lines as meridians and transfer the direction information of the north arrow to the meridians and establish precedence relations for locations on them.

The second possibility provides a more general solution that is able to cope with curved grid lines in the same way as with straight grid lines. Nevertheless we have to assume that the north arrow distinguishes meridians from parallels of latitude. We first take two parallels of latitude into account. The one that passes the tip of the north arrow and the one that passes the end of the north arrow. The arrow specifies then the parallel of latitude further north that is the one passing the tip. With two ordered parallels of latitude we can establish ordering information necessary for introducing qualitative geographic coordinates.

3 Formal Characterization of Grids in Maps

According to section 2 the formal description of grids as reference systems is based on geometry since this allows the coding of ordering information. The proposed geometric framework enables a description of grid-based reference systems in maps. The focus of the analysis lies on spatial relations that are specified through cardinal directions. We turn our attention to the question which assumptions have to be made to decide if an object is further north (south, east or west) than another object.

For the grid reference system on the earth holds that every point is completely specified by one meridian and one parallel of latitude, and more important through every point there is one uniquely determined meridian and one parallel of latitude. For projections it is generally not possible to determine the corresponding grid lines in a map to a given point. The map has to visualize at least some reference grid lines to generate uniquely determined grid lines through a given point.

The geometric description relates at first points and grid lines, and subsequently introduces the ordering structure. The points represent positions of objects and the grid lines characterize the meridians and parallels of latitude.

3.1 Geometric Structure

The basic geometric entities in our approach are points and grid lines on a given map (area). Capitals in italics like P, P', P_1, ..., Q and R denote points and lower case italics like g, g', g_1, g_2, ... denote grid lines. The map is denoted by μ. The axiomatic characterization models their dependencies by relating them through axioms. This is done via three basic relations: incidence (ι), quasi-parallelity ($\wr\wr$) and betweenness (β). The geometric approach gives no definition what a point or grid line actually "is", but it specifies their properties through a list of axioms that they satisfy. Since we aim at the modeling of map projections we take the axioms of planar geometry as a basis.

The incidence structure relates points and grid lines on a map. A point is incident with a grid line, if it lies on the grid line or the grid line goes through the point. The incidence axioms specify the properties of points depending on grid lines and vice versa. Basically, they are founded on the ordinary axioms for points and straight lines since grid lines have many properties with straight lines in common. But we emphasize that grid lines can, in principle, be arbitrarily curved. Details on axioms for points and straight lines can be found in Eschenbach & Kulik (1997).

3.2 Axioms of Incidence

The following axioms record five basic properties for systems of grid lines in maps. These axioms relate points and grid lines on a given map μ using the incidence relation ι. According to axiom (I1) for every grid line there are at least two different points that are incident with this line. Axiom (I2) says that grid lines are uniquely determined by two points. This implies that two different grid lines intersect at most once. Axiom (I3) guarantees that there are exactly two different grid lines through any given point on a map. This axiom is characteristic for grid structures as the graticule. The fourth axiom (I4) ensures that the underlying map structure is at least planar, i.e., not all points of a map are incident with a single grid line. Axiom (I5) guarantees that grid lines are completely included in a map. One simple consequence is that every point lies on the given map μ.

(I1) $\forall g\ \exists P\ Q \quad [P \neq Q \wedge P\ \iota\ g \wedge Q\ \iota\ g]$

(I2) $\forall P\ Q\ g\ g' \quad [P \iota g \wedge Q \iota g \wedge P \iota g' \wedge Q \iota g' \wedge P \neq Q \Rightarrow g = g']$

(I3) $\forall P \quad [P \iota \mu \Rightarrow$
$\exists g\ g'\ [g \neq g' \wedge P \iota g \wedge P \iota g' \wedge \forall g''\ [P \iota g'' \Rightarrow g'' = g \vee g'' = g']]]$

(I4) $\forall g\ \exists P \quad [\neg(P \iota g) \wedge P \iota \mu]$

(I5) $\forall g\ P \quad [P \iota g \Rightarrow P \iota \mu]$

The incidence axioms for grid structures differ from the incidence axioms used by Hilbert (1956) for points and straight lines. For example, the axiom requiring that any two points lie on a common straight line does not hold for grid lines since in general it is not possible to connect any two points with a grid line.

3.3 Axioms of Quasi-Parallelity

An essential relation between straight lines in Euclidean Geometry is parallelity. Two straight lines are parallel iff they are either equal or do not have any point in common. According to the Euclidean axiom, for every straight line and every point there is a uniquely determined parallel straight line through this point. We want to characterize an analogous concept to parallelity to distinguish two classes of grid lines, those representing meridians and those representing the parallels of latitude. But the definition of parallelity for straight lines cannot be applied to grid lines since there may be grid lines representing meridians and those representing parallels of latitude that do not share a point. Hence, they would be parallel according to the above definition. We therefore introduce the relation of quasi-parallelity which is symbolized as $\wr\wr$.

The properties of this relation are characterized as follows. According to axiom (Q1) for every grid line and every point there is a quasi-parallel grid line through this point. Axiom (Q2) says that any two quasi-parallel grid lines are equal or do not have any common point. The third axiom (Q3) states that quasi-parallelity is transitive.

(Q1) $\forall g\ P\ \exists g'$ $\quad [P\ \iota\ g' \wedge g \wr\wr g']$

(Q2) $\forall g\ g'$ $\quad [g \wr\wr g' \Rightarrow g = g' \vee \neg\exists P\ [P\ \iota\ g \wedge P\ \iota\ g']]$

(Q3) $\forall g\ g'\ g''$ $\quad [g \wr\wr g' \wedge g' \wr\wr g'' \Rightarrow g \wr\wr g'']$

We mention some consequences of these axioms. First, quasi-parallelity is an equivalence relation. The uniqueness of quasi-parallels through a given point relative to a given grid line is ensured by Theorem (T1). Theorem (T2) states that two different grid lines that are intersected by a third grid line are quasi-parallel. According to Theorem (T3) grid lines can be subdivided in two classes, since every grid line is quasi-parallel to one of two given intersecting grid lines. Hence, grid lines constitute two equivalence classes with respect to quasi-parallelity.

(T1) $\forall g'\ g''$ $\quad [\exists g\ P\ [P\ \iota\ g' \wedge P\ \iota\ g'' \wedge g \wr\wr g' \wedge g \wr\wr g''] \Rightarrow g' = g'']$

(T2) $\forall g\ g'$ $\quad [\exists g''\ P\ Q\ [P\ \iota\ g \wedge P\ \iota\ g'' \wedge Q\ \iota\ g' \wedge Q\ \iota\ g'' \wedge g \neq g'' \wedge g' \neq g''] \Rightarrow g \wr\wr g']$

(T3) $\forall g\ g'$ $\quad [g \neq g' \wedge \exists P\ [P\ \iota\ g \wedge P\ \iota\ g'] \Rightarrow \forall g''\ [g'' \wr\wr g \vee g'' \wr\wr g']]$

Theorem (T3) reflects the fact that every grid in a map consists of meridians and parallels of latitude. The developed geometry is able to characterize all grids resulting from projections which do neither include the North Pole nor the South Pole. That is to say the meridians do not meet in the map. This is reflected in the geometry since grid lines of the same equivalence class do not meet either. There are exactly two equivalence classes, one representing the meridians and one representing the parallels of latitude. In these grids there is no difference for reasoning about cardinal directions concerning north/south and west/east since both classes of grid lines have the same spatial properties and hence are characterized in the same way.

3.4 The Ordering Structure

In the next step we introduce the ordering structure via axioms of betweenness. The betweenness structure in essence relates (at least) three points on a grid line (cf. Eschenbach, Habel & Kulik, 1999, Huntington, 1938). We characterize the betweenness properties of grid lines first and define on this basis the betweenness structure of points on a grid line.

For matters of convenience, we introduce the following definition. Three pairwise distinct and quasi-parallel grid lines are called a quasi-parallel triple.

$$\mathrm{qpt}(g_1, g_2, g_3) \quad \Leftrightarrow_{\mathrm{def}} \quad g_1 \wr\wr g_2 \wedge g_1 \wr\wr g_3 \wedge g_2 \wr\wr g_3 \wedge g_1 \neq g_2 \wedge g_1 \neq g_3 \wedge g_2 \neq g_3$$

Let g_1, g_2 and g_3 denote grid lines. The betweenness relation for grid lines has to fulfill the following conditions: (β1) If g_2 is between g_1 and g_3, then g_1, g_2 and g_3 are a quasi-parallel triple. Axiom (β2) expresses the symmetry of betweenness of grid lines with respect to the first and the third argument: If g_2 is between g_1 and g_3 then g_2 is between g_3 and g_1. Axiom (β3) ensures that at most one of three grid lines is between the other two. Axiom (β4) states that for a quasi-parallel triple of grid lines at least one of them is between the other two. Therefore, betweenness of grid lines constitutes a total order. The last axioms (β5) expresses that if g_2 is between g_1 and g_3 and g another grid line distinct from g_2 then g_2 is either between g_1 and g or between g and g_3.

(β1) $\forall g_1\, g_2\, g_3 \quad [\beta(g_1, g_2, g_3) \Rightarrow \mathsf{qpt}(g_1, g_2, g_3)]$

(β2) $\forall g_1\, g_2\, g_3 \quad [\beta(g_1, g_2, g_3) \Rightarrow \beta(g_3, g_2, g_1)]$

(β3) $\forall g_1\, g_2\, g_3 \quad [\beta(g_1, g_2, g_3) \Rightarrow \neg\beta(g_2, g_1, g_3)]$

(β4) $\forall g_1\, g_2\, g_3 \quad [\mathsf{qpt}(g_1, g_2, g_3) \Rightarrow \beta(g_1, g_2, g_3) \vee \beta(g_2, g_1, g_3) \vee \beta(g_1, g_3, g_2)]$

(β5) $\forall g_1\, g_2\, g_3 \quad [\beta(g_1, g_2, g_3) \wedge g \wr\wr g_2 \wedge g \neq g_2 \Rightarrow \beta(g_1, g_2, g) \vee \beta(g, g_2, g_3)]$

The betweenness structure of grid lines can be used to induce a betweenness structure for points on a given grid line. Three points P, Q and R are *collinear* iff the points are pairwise distinct and lie on a single grid line g.

$$\mathsf{col}(P_1, P_2, P_3) \quad \Leftrightarrow_{\text{def}} \quad P_1 \neq P_2 \wedge P_1 \neq P_3 \wedge P_2 \neq P_3 \wedge \exists g\, [P_1\, \iota\, g \wedge P_2\, \iota\, g \wedge P_3\, \iota\, g]$$

A point P_2 is between two other points P_1 and P_3 iff the three points are collinear and a grid line through P_2 is between two grid lines through P_1 and P_3.

$$\beta(P_1, P_2, P_3) \quad \Leftrightarrow_{\text{def}} \quad \mathsf{col}(P_1, P_2, P_3) \wedge \exists g_1\, g_2\, g_3\, [\beta(g_1, g_2, g_3) \wedge P_1\, \iota\, g_1 \wedge P_2\, \iota\, g_2 \wedge P_3\, \iota\, g_3]$$

Employing this definition it can be shown that all expected properties for points on grid lines are fulfilled: Betweenness for points on grid lines is symmetric with respect to the first and third argument, exactly one point lies between two other points and every point divides a grid line into two halves (cf., Eschenbach et al., 1998). The following theorem expresses that points on two different quasi-parallel grid lines are ordered in the same way (see Fig. 2).

(T4) $\forall g_1\, g_2\, g_3\, P\, Q\, R\, P'\, Q'\, R'\, [\beta(P, Q, R) \wedge P\, \iota\, g_1 \wedge Q\, \iota\, g_2 \wedge R\, \iota\, g_3 \wedge P'\, \iota\, g_1 \wedge Q'\, \iota\, g_2 \wedge R'\, \iota\, g_3 \wedge \mathsf{col}(P', Q', R') \Rightarrow \beta(P', Q', R')]$

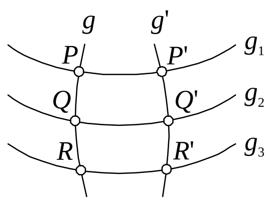


Fig. 2. If the intersection point of g and g_2 is between the intersection points of g and g_1 and of g and g_3 then the intersection point of g' and g_2 is between the intersection points of g' and g_1 and of g' and g_3

The betweenness axioms for grid lines enable us to describe that two points on a map are on the same side or on different sides of a grid line. Two points are on *different sides* with respect to a given grid line g, if g is between the grid lines which pass through the points, and they are on the *same* side, if g not between the quasi-parallel grid lines passing through the points.

$$\delta(g, P, Q) \quad \Leftrightarrow_{def} \quad \exists g_1\, g_2\, [P\,\iota\, g_1 \wedge Q\,\iota\, g_2 \wedge \beta(g_1, g, g_2)]$$

$$\sigma(g, P, Q) \quad \Leftrightarrow_{def} \quad \neg(P\,\iota\, g \vee Q\,\iota\, g) \wedge \forall g_1\, g_2\, [P\,\iota\, g_1 \wedge Q\,\iota\, g_2 \wedge g_1 \wr\wr g \wedge g \wr\wr g_2 \Rightarrow \neg\beta(g_1, g, g_2)]$$

In the next step we define the notion of an *endpoint* of a grid line.

$$\mathsf{ept}(P, g) \quad \Leftrightarrow_{def} \quad P\,\iota\, g \wedge \forall Q\, R\, [Q\,\iota\, g \wedge R\,\iota\, g \Rightarrow \neg\beta(Q, P, R)]$$

A point is a *map boundary point* iff it is an endpoint of any grid line.

$$\mathsf{mb}(P) \quad \Leftrightarrow_{def} \quad \exists g\, [\mathsf{ept}(P, g)]$$

Axiom (β6) requires that every grid line has (at least) two endpoints.

(β6) $\forall g\, \exists P\, Q \quad [P \neq Q \wedge \mathsf{ept}(P, g) \wedge \mathsf{ept}(Q, g)]$

Since exactly one point lies between two other points it follows that there are at most two endpoints of a grid line. Hence, every grid line has exactly two endpoints.

3.5 Qualitative Geographic Coordinates

To localize objects in maps geographers usually employ coordinates of real numbers. In order to reason about cardinal directions it is not necessary to presuppose precise assignment of numbers. Even if the exact geographic coordinates are not known inferences concerning the question whether one place is further north than another place are still possible. Qualitative geographic coordinates are suitable for inferences of this kind. They do not involve any conception of numbers since they can be defined based on ordering information. Therefore, we omit numbers as coordinates and use grid lines instead. Every grid line represents a coordinate and we can associate two grid lines with every point standing for a meridian and a parallel of latitude. The grid lines can be ordered and this information is used for reasoning about cardinal directions.

In order to distinguish a grid line as a meridian or a latitude we need at least two points (see axiom (I2)). There are several possibilities to induce two reference points. We basically consider two alternatives: The case of maps supplied with a north arrow and the case of maps with a boundary consisting of grid lines where one of the sides is distinguished as the geographic north. In the first case the north arrow distinguishes one equivalence class of grid lines as meridians (see section 2.2). The tip and the end of the north arrow intersects two parallels of latitude g_T and g_E respectively and orders them such that g_T is further north than g_E. We can employ the ordering information of the north arrow to associate with every map a grid line which is the northernmost one. This grid line is called the north grid line ng of a map. It is not between any two other grid lines and orders the grid lines g_T and g_E such that g_T is between ng and g_E.

$$g = \mathsf{ng} \quad \Leftrightarrow_{def} \quad \beta(g, g_T, g_E) \wedge \forall g_1\, g_2\, [\neg\beta(g_1, g, g_2)]$$

In the second case the boundary of the map consists of four grid lines. The side distinguished as geographic north only has to fulfill the second part of the definition of the north grid line and the grid lines running from the opposite side of the geographic north to the north grid line are the meridians. Summarizing both cases we require in the next axiom that every map has to include a north grid line.

(β7) $\exists g \quad [g = \mathsf{ng}]$

Using the north grid line we can give the definitions of a parallel of latitude (denoted by lat) and a meridian (symbolized as mer). A grid line is a parallel of latitude iff it is quasi-parallel to the north grid line. Otherwise, it is a meridian.

$$\mathsf{lat}(g) \quad \Leftrightarrow_{\text{def}} \quad g \wr\wr \mathsf{ng}$$

$$\mathsf{mer}(g) \quad \Leftrightarrow_{\text{def}} \quad \neg\mathsf{lat}(g)$$

With these preparations we can determine, whether one parallel of latitude precedes another parallel of latitude and whether one parallel of latitude is preceded by another parallel of latitude.

$$\prec_{\mathsf{lat}}(g_1, g_2) \quad \Leftrightarrow_{\text{def}} \quad \mathsf{lat}(g_1) \wedge \mathsf{lat}(g_2) \wedge \beta(\mathsf{ng}, g_1, g_2)$$

$$\succ_{\mathsf{lat}}(g_1, g_2) \quad \Leftrightarrow_{\text{def}} \quad \prec_{\mathsf{lat}}(g_2, g_1).$$

If a map is supplied with a north arrow then it is known by convention that a meridian lying in the left half induced by the arrow is further west than a meridian lying in the right half. Therefore we can analogously define the westernmost grid line of a map called west grid line and denoted by wg. For a map (without a north arrow) with a boundary consisting of grid lines we can define the side distinguished as geographic west as the west grid line just as in the case of the north grid line. Similar, we have to require then that every map has to contain a west grid line. We omit the details since we focus on the north–/south relation. One meridian is further west (east) than another meridian iff the following holds:

$$\prec_{\mathsf{mer}}(g_1, g_2) \quad \Leftrightarrow_{\text{def}} \quad \mathsf{mer}(g_1) \wedge \mathsf{mer}(g_2) \wedge \beta(\mathsf{wg}, g_1, g_2)$$

$$\succ_{\mathsf{mer}}(g_1, g_2) \quad \Leftrightarrow_{\text{def}} \quad \prec_{\mathsf{mer}}(g_2, g_1).$$

4 Geometric Structure of Maps

On the basis of the geometric inventory we describe in this section how grids and qualitative geographic coordinates can be employed to reason about spatial relations involving cardinal directions. First, we specify the properties a map has to fulfill so that it can be treated in this framework. The basic task then is to deduce the spatial relations expressed via cardinal directions of two points on a map. We present two alternative characterizations. In section 4.2 we use grid lines dividing the map area into two halves. Section 4.3 shows how to use grid lines as qualitative geographic coordinates and—more specifically—the ordering structure of the grid lines.

4.1 On Map Parts

Not every map with a graticule is a map in our technical sense. As already mentioned, the map has to be able to distinguish one class of grid lines as meridians and the other one as parallels of latitude and it has to supply a precedence relation for grid lines. These two requirements can be met in two independent ways: A map has to contain either a north arrow or its boundary is given by grid lines where one of the lines is distinguished as geographic north.

Since a map only depicts a part of the earth's surface we note that in general the north grid line would consist of only one point. The required north grid line only touches the map at its boundary (see Fig. 3). Therefore, the formal map has to exclude these grid lines which are degenerated to point in the map (see Fig. 3).

Fig. 3. Two Maps: The left map cannot contain a north grid line since the possible candidate for the north grid line shares only a single point with the map. The right map is a corresponding formal map including a north grid line

In order to omit an unnecessary reduction of the initial map we choose an adaptive way to construct the formal map. Suppose we want to determine the spatial relation of two points using cardinal directions. Then we consider a smaller map containing the two points and fulfilling the conditions given in section 3 such that the formal map in particular has a north grid line (cf. Fig. 3 again).

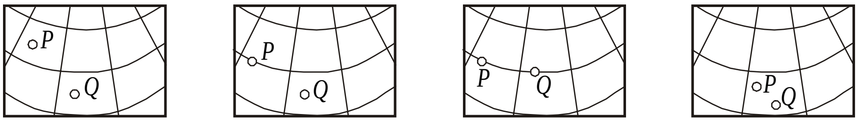
However, in many cases we can consider the map as a whole since every map possessing a boundary given by grid lines is a formal map in our sense. For the following, we assume that the grid lines of the underlying map satisfy the axioms of section 3.

4.2 Cardinal Directions Characterized by Grid Lines

Not every grid line is visible on a map. A graticule just highlights some grid lines by visualizing them. We call such distinguished grid lines *visualized grid lines*. Visualized grid lines allow simplified inferences about cardinal directions on a map.

To determine if a point P is further north than another point Q we have to distinguish two essentially different cases: (I) The two points are related with respect to a visualized grid line representing a parallel of latitude (abbreviated as visualized latitude) or (II) both points belong to a single grid cell. The first case (I) divides up into three subcases: (1) the points lie in different sectors of the grid which are divided by at least one visualized latitude, (2) one of the points lies on a visualized latitude, but not the other, (3) both points lie on the same visualized latitude.

1. In this case the visualized latitude divides the map into two parts: If *P* is on the same side as a point on the north grid line—abbreviated as north point in the following—then *Q* is on the other side as the north point and therefore *P* is further north than *Q*. Otherwise, if *Q* is on the same side as the north point then *Q* is further north than *P* or *P* further south than *Q*.
2. If one of the two points is on a visualized latitude, for example *P*, then if *Q* is on the same side as the north point, *Q* is further north than *P*, otherwise *P* is further north than *Q*.
3. If both points are on the same visualized latitude then both points are on the same level and no one is further north than the other is.



I a I b I c II

Fig. 4. There are four different cases for the spatial relations of two points dependent on visualized grid lines representing the parallels of latitude

In case (II) we have to construct a further grid line. We demonstrate the method just for point *P*. According to axiom (I3) there are two grid lines through *P*. It might not be an easy task in a real map to construct these grid lines, even if it is in general possible. But at least in some important cases of the cylindrical and conical projections there is an easy geometric condition that specifies the grid lines. Using the transitivity of quasi-parallelity and theorem (T3) we obtain that one of these two grid lines is also a parallel of latitude. Now, there are three alternatives for *Q* as in case I. *Q* is also on this line, then *P* and *Q* are on the same level. If *Q* is on the same side as a point on the north grid line then *Q* is further north than *P* and in the remaining case where *Q* and this point are on different sides *P* is further north than *Q*.

We omit the formal description of case (I) where a visualized latitude *g* is used to determine the relation 'further north than'. The characterization is similar to the more general case (II) where one has to construct a grid line representing a parallel of latitude. The characterization of case (II) for the relations 'further north than' (symbolized as north), 'further south than' (symbolized as south) and 'being on the same latitude' (symbolized as slt) has a simple form.

$$\text{north}(P, Q) \quad \Leftrightarrow_{\text{def}} \quad \exists g\, R\, [R\, \iota\, \text{ng} \wedge \text{lat}(g) \wedge P\, \iota\, g \wedge \delta(g, Q, R)]$$

$$\text{south}(P, Q) \quad \Leftrightarrow_{\text{def}} \quad \exists g\, R\, [R\, \iota\, \text{ng} \wedge \text{lat}(g) \wedge P\, \iota\, g \wedge \sigma(g, Q, R)]$$

$$\text{slt}(P, Q) \quad \Leftrightarrow_{\text{def}} \quad \exists g\, [\text{lat}(g) \wedge P\, \iota\, g \wedge Q\, \iota\, g]$$

The characterizations of 'further west than' and 'further east than' are similar and are therefore omitted (the differences limit to the replacement of lat by mer and ng by wg).

4.3 Cardinal Directions Characterized with Qualitative Geographic Coordinates

In this section we use grid lines as qualitative geographic coordinates for inferences about cardinal directions. The qualitative geographic coordinates enable a much simpler description for cardinal directions. A point P is further north than a point Q iff the qualitative geographic coordinate representing the parallel of latitude through P precedes the qualitative geographic coordinate representing the parallel of latitude through Q. We denote the uniquely determined qualitative geographic coordinate representing the parallel of latitude by cl and the one representing the meridian by cm:

$$\mathsf{cl}(P) = g \quad \Leftrightarrow_{\text{def}} \quad P\,\iota\,g \wedge \mathsf{lat}(g)$$

$$\mathsf{cm}(P) = g \quad \Leftrightarrow_{\text{def}} \quad P\,\iota\,g \wedge \mathsf{mer}(g)$$

Then, we get for the relations ‘further north than’, ‘further south than’ and ‘being on the same latitude’

$$\mathsf{north}(P, Q) \quad \Leftrightarrow_{\text{def}} \quad \prec_{\mathsf{lat}}(\mathsf{cl}(P), \mathsf{cl}(Q))$$

$$\mathsf{south}(P, Q) \quad \Leftrightarrow_{\text{def}} \quad \succ_{\mathsf{lat}}(\mathsf{cl}(P), \mathsf{cl}(Q))$$

$$\mathsf{slt}(P, Q) \quad \Leftrightarrow_{\text{def}} \quad \mathsf{cl}(P) = \mathsf{cl}(Q)$$

The corresponding relations ‘further west than’, ‘further east than’ and ‘being on the same meridian’ are characterized as follows

$$\mathsf{west}(P, Q) \quad \Leftrightarrow_{\text{def}} \quad \prec_{\mathsf{mer}}(\mathsf{cm}(P), \mathsf{cm}(Q))$$

$$\mathsf{east}(P, Q) \quad \Leftrightarrow_{\text{def}} \quad \succ_{\mathsf{mer}}(\mathsf{cm}(P), \mathsf{cm}(Q))$$

$$\mathsf{smd}(P, Q) \quad \Leftrightarrow_{\text{def}} \quad \mathsf{cm}(P) = \mathsf{cm}(Q)$$

4.4 Inferences on Qualitative Geographic Coordinates

The three defining relations used for the characterizations in section 4.3 are $\prec$, $\succ$ and $=$. These relations correspond to the ones used in the analysis given by Ligozat (1998). Ligozat assumes that for a given point there are nine basic relations for another point with respect to cardinal directions when using a system of two axes as reference system. For a fixed point P another point Q can be equal with P, or be north, north-east, east, ... of P. These relations are noted as *eq*, *n*, *ne*, *e*, ..., respectively. On these relations the operation of composition can be defined: Given three points P, Q and R, where the relations between P and Q as well as between Q and R are known, what is the relation between P and R then? The answers are recorded in a composition table. Ligozat (1998) has given a composition table for the nine basic relations bases on cardinal directions. These relations can be translated in our calculus. We demonstrate this for two examples *n* and *ne*:

$$P\ n\ Q \quad \Leftrightarrow_{\text{def}} \quad \mathsf{north}(P, Q) \wedge \mathsf{smd}(P, Q)$$

$$P\ ne\ Q \quad \Leftrightarrow_{\text{def}} \quad \mathsf{north}(P, Q) \wedge \mathsf{east}(P, Q)$$

Since these relations can be translated, our geometric framework presented here can be considered as a geometric basis of the composition table of Ligozat. Within this calculus all the statements of the composition table can be proven. This shows in addition that no conception of orthogonality or straightness of the two axes is required for the inferences since the geometry does not employ any conception of angles or straightness. The only assumption is that the system of the two axes can be supplied in a unique way with a grid. Moreover, it turns out that the composition table of Ligozat holds in general as long as the spatial structure can be supplied with any kind of a grid in our sense.

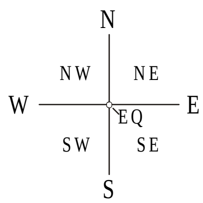


Fig. 5. Nine basic relations of one point with respect to another point based on cardinal directions

5 Conclusion and Outlook

Our analysis proposes a qualitative calculus for spatial reasoning on cardinal directions. We have stressed that different grid structures resulting from mappings of the surface of the earth to a planar medium can never preserve information about angles, distances and areas at the same time. Nonetheless, the resulting maps provide enough information to reason about cardinal relations between two cartographic objects. It turns out that ordering geometry and the calculus of qualitative coordinates provide a sufficient basis to describe the inferences using cardinal directions. The grid supplied with a north arrow provides an absolute reference system and preserves the ordering information of the meridians and the parallels of latitude.

This paper introduced two reasoning strategies. The first strategy uses grid lines to divide the map area into two halves and determines for two objects which object position is on the same side as a point on the north grid line. The second strategy introduces grid lines as qualitative geographic coordinates and employs their ordering structure to decide for two objects which one is further north. The calculus provides a geometric basis for existing approaches like the one of Ligozat (1998) which treat cardinal directions in the domain of qualitative reasoning.

The following objectives guide our future work. We intend to extend the calculus so that it enables inferences about cardinal directions on map projections including the poles. Since our approach can be employed for other grid structures involving ordering information, we look for possible applications in GIS of general grid structures that satisfy the presented axioms but are not given by meridians and parallels of latitude.

Grids constitute absolute reference systems coding ordering information. There are other types of reference systems in maps given for example by half planes or sectors (cf. Hernández, 1992). It the next step we evaluate if these reference systems also provide essentially ordering information or if we have to enrich the geometric framework to describe them.

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